

Coherent states for the hydrogen atom: discrete and continuous spectra

[Coherent states for the hydrogen atom]

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Abstract. We construct the systems of generalised coherent states for the discrete and continuous spectra of the hydrogen atom. These systems are expressed in terms of elementary functions and are invariant under the $SO(3, 2)$ (discrete spectrum) and $SO(4, 1)$ (continuous spectrum) subgroups of the dynamical symmetry group $SO(4, 2)$ of the hydrogen atom. Both systems of coherent states are particular cases of the kernel of integral operator which intertwines irreducible representations of the $SO(4, 2)$ group.

1. Introduction

The problem of constructing the generalised coherent states (CS) for the hydrogen atom was first formulated by Schrodinger in 1926 simultaneously with the construction of CS for the harmonic oscillator (HO). Since then, there have been attempts to solve this problem on the basis of the Kustaanheimo–Stiefel transformation connecting 4D HO and 3D HA [1, 2]. Within this approach the CS for 4D HO are constructed, whereas the CS for 3D HA are obtained from the above CS either using constraints imposed on a set of parameters [1] or by integration over the additional variable [2]. However, the packets obtained in [2] spread with time; even within the time when the packets preserve their shape, they can be considered as simulating the Kepler motion only for large quantum numbers and, as it was shown in [3], when restricted to a plane. The same statements are also valid for the states constructed in [1].

It is also possible (as it was suggested in [4]) to start from the existence of integrals of motion. Due to the dynamical symmetry group $SO(4, 2)$ of the HA [5], this idea naturally leads to the construction of CS by the Barut–Girardello method, as it was performed in [6]. Such states evolve consistently under the pseudo-Hamiltonian $(-2H)^{-1/2}$; however, during evolution in real time they spread and cannot be expressed in the closed form in configuration space. Much progress has been achieved in constructing the CS for the radial Schrodinger equation for the HA [7, 8]. These states are expressed in terms of elementary functions of \mathbf{r} , are $SO(2, 1)$ -invariant, and minimize the uncertainty relation for the suitably defined operators \mathbf{X} and \mathbf{P} .

Thus, the attempts to establish the quantum–classical correspondence for the 3D HA, using the HO as a model, failed. In fact, an extremely simple form of this correspondence for the HO is a consequence of the fact that the HO energy levels are equidistant; as a result, such simple correspondence cannot exist for the HA. This circumstance has been recently mentioned in [9]; however, this reasoning can be found even in one of N. Bohr’s fundamental papers on quantum mechanics [10] (with reference to Heisenberg and Darwin).

Therefore, in our opinion, the formulation of quantum–classical correspondence for the HA should be based on mathematical principles rather than on dynamical considerations. Recently, two approaches have been proposed in this direction. The first approach suggested by Klauder [11] is based on his continuous-representation theory [12] and later was developed in a number of papers (see [13] and references therein). This approach implies that the following requirements should be satisfied by the CS system: (i) the dependence on the parameters is continuous, (ii) the resolution of the identity takes place, and (iii) the Hamiltonian yields the evolution in parameter space. The disadvantages of this approach include the infiniteness of the evolution in the parameter which is inconsistent with periodicity of the corresponding classical motion, and the absence of a closed-form expression for the CS in configuration space.

group and the corresponding $SO(3, 2)/(SO(3) \otimes SO(2))$ symmetric space. In section 3.2 a CS system for the discrete spectrum is constructed; CS for the 1D HA constructed in [8] are the particular case of our CS. In section 3.3 we show that this system is, in fact, a $SO(3, 2)$ -invariant system of the Perelomov's CS. In section 2.4 we show that this CS system minimizes the so-called Robertson inequality for four-dimensional coordinates and momenta; this inequality is a generalisation of the Heisenberg uncertainty relation to the case of n variables. In this respect the constructed CS system is similar to the usual CS for the Heisenberg–Weyl group. In section 4 the continuous spectrum is considered, namely, in section 4.1 we consider the $SO(4, 1)$ group and its action on \mathbb{R}^3 and in section 4.2 the wave functions of the continuous spectrum of HA are considered. In section 4.3 we follow the ideas of [25] concerning the Mellin transform for the hypergeometric functions to construct the CS system for the continuous HA spectrum; this system is similar to that constructed in section 3.2. In section 5 it is shown that CS systems for the discrete and continuous spectra of HA are particular cases of a function which intertwines the different irreducible representations of the $SO(4, 2)$ group (for more details on this group and its representations see [26, 27] and references therein). We also establish a relation between the results obtained above and that reported recently [28] and concerned with representation of the HA wave functions in terms of the classical motion.

2. Preliminaries

2.1. The Kustaanheimo-Stiefel transformation

Let us introduce the four-vector

$$n_{\mathcal{X}}^{\mu} = (r, \mathbf{x}) \quad n_{\mathcal{X}} \cdot n_{\mathcal{X}} = 0 \quad n_{\mathcal{X}}^0 \geq 0 \quad \mu, \nu, \dots = 0, \dots, 3.$$

Denote the Cartesian coordinates in \mathbb{R}^4 as $\xi_{\alpha}, \eta_{\alpha}$, $\alpha, \beta = 1, 2$. We also need two other coordinate systems in \mathbb{R}^4 : $\xi, \eta, \varphi_{\xi}, \varphi_{\eta}$ and the complex two-dimensional coordinates defined by

$$\begin{aligned} z_1 &= \frac{1}{\sqrt{2}}(\xi_1 + i\xi_2) = \frac{1}{\sqrt{2}}e^{i\varphi_{\xi}} \\ z_2 &= \frac{1}{\sqrt{2}}(\eta_1 + i\eta_2) = \frac{1}{\sqrt{2}}e^{i\varphi_{\eta}}. \end{aligned} \tag{1}$$

Then the Kustaanheimo-Stiefel transformation takes the form

$$n_{\mathcal{X}}^{\mu} = r_0 \bar{z}_{\alpha} (\sigma^{\mu})_{\alpha\beta} z_{\beta} \quad \sigma^{\mu} = (1, \boldsymbol{\sigma}).$$

Then the Schrödinger equation for the 3D

$$\left(-\frac{\hbar^2}{2\mu} \frac{\partial^2}{\partial \mathcal{X}^2} - \frac{e^2}{r} \right) \psi = E\psi \tag{2}$$

may be rewritten in the four-dimensional form as

$$\left[\left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial \xi_{\alpha} \partial \xi_{\alpha}} + \frac{1}{2} m \omega^2 \xi_{\alpha} \xi_{\alpha} \right) + (\xi \rightarrow \eta) \right] \psi = \varepsilon \psi \tag{3}$$

$$\left(\frac{\partial}{\partial \varphi_{\xi}} - \frac{\partial}{\partial \varphi_{\eta}} \right) \psi = 0 \tag{4}$$

where the following notations are introduced:

$$m = 4\mu \quad \varepsilon = \frac{e^2}{2r_0} \quad r_0 = \hbar^2/(me^2) \quad \omega = (-E/2\mu)^{1/2}.$$

Let a solution $\psi = \psi(\xi, \eta, \varphi_{\xi}, \varphi_{\eta})$ of the equations (3),(4) is known, then we can obtain the solution of equation (2) setting $\varphi_{\xi} = \varphi_{\eta} = \varphi/2$.

Then the Kustaanheimo-Stiefel transformation reduces to usual transformation to the parabolic coordinates

$$\begin{aligned} x_1 &= r_0 \xi \eta \cos \varphi & x_2 &= r_0 \xi \eta \sin \varphi \\ x_3 &= \frac{r_0}{2}(\xi^2 - \eta^2) & r &= \frac{r_0}{2}(\xi^2 + \eta^2). \end{aligned}$$

Let $E < 0$; then using restretching of coordinates we can reduce equations (3),(4) to two Schrodinger equations for two 2D HO with unit mass and frequency and with the same values of angular momentum. Then the functions

$$\begin{aligned}\Psi_{\mathbf{n}_1 \mathbf{n}_2 \mathbf{m}}(\mathcal{X}) &= \psi_{\mathbf{n}_1 \mathbf{n}_2 \mathbf{m}}(\mathcal{X}/(nr_0)) \\ \mathbf{n} = \mathbf{n}_1 + \mathbf{n}_2 + |\mathbf{m}| + 1 \quad E &= -\frac{1}{2n^2}\end{aligned}$$

obey equation (2), where

$$\psi_{\mathbf{n}_1 \mathbf{n}_2 \mathbf{m}}(\mathcal{X}) \equiv |n_1 n_2 m\rangle = (-1)^{n_1 + \frac{1}{2}(m - |m|)} \frac{e^{im\phi}}{\pi^{1/2}} \varphi_{n_1 |m|}(\xi) \varphi_{n_2 |m|}(\eta)$$

and

$$\varphi_{n|m|}(\xi) = e^{-\xi^2/2} \xi^{|m|} \left(\frac{n!}{(n + |m|)!} \right)^{1/2} L_n^{|m|}(\xi^2)$$

is a solution of the radial Schrodinger equation for the 2D HO with unit mass and frequency and with the angular momentum equal to $|m|$. The above solutions are real and normalized as

$$\int_0^\infty d(\xi^2) \varphi_{n|m|}(\xi) \varphi_{n'|m|}(\xi) = \delta_{nn'}.$$

Then taking into account that $r^{-1} d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3 = d(\xi^2) d(\eta^2) d\varphi$ we obtain

$$\langle n_1 n_2 m | n'_1 n'_2 m' \rangle \equiv \int_{\mathbb{R}^3} r^{-1} d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3 \psi_{n_1 n_2 \mathbf{m}}(\mathcal{X}) \psi_{n'_1 n'_2 \mathbf{m}'}(\mathcal{X}) = \delta_{n_1 n'_1} \delta_{n_2 n'_2} \delta_{m m'}. \quad (5)$$

The measure in the right-hand side of the above expression is just a Lorentz-invariant measure over the light cone $\mathbf{n}\mathbf{x} \cdot \mathbf{n}\mathbf{x} = 0$.

2.2. The dynamical symmetry group

Let us introduce the operators a_α, b_α and their Hermitian conjugates as

$$\begin{aligned}a_\alpha &= \frac{1}{\sqrt{2}}(\bar{z}_\alpha + \partial_{z_\alpha}) & a_\alpha^\dagger &= \frac{1}{\sqrt{2}}(z_\alpha - \partial_{\bar{z}_\alpha}) \\ b_\alpha &= \frac{1}{\sqrt{2}}\epsilon_{\alpha\beta}(z_\beta + \partial_{\bar{z}_\beta}) & b_\alpha^\dagger &= \frac{1}{\sqrt{2}}\epsilon_{\alpha\beta}(\bar{z}_\beta - \partial_{z_\beta}).\end{aligned} \quad (6)$$

Then nonvanishing commutation relations are

$$[a_\alpha, a_\beta^\dagger] = [b_\alpha, b_\beta^\dagger] = \delta_{\alpha\beta}.$$

Then we can write the vectors $|n_1 n_2 \mathbf{m}\rangle$ as

$$\begin{aligned}|n_1 n_2 \mathbf{m}\rangle &= [n_1!(n_1 + |m|)!n_2!(n_2 + |m|)!]^{-1/2} \\ &\times \begin{cases} a_1^{\dagger n_2 + |m|} a_2^{\dagger n_1} b_1^{\dagger n_1 + |m|} b_2^{\dagger n_2} |0\rangle & m \geq 0 \\ a_1^{\dagger n_2} a_2^{\dagger n_1 + |m|} b_1^{\dagger n_1} b_2^{\dagger n_2 + |m|} |0\rangle & m \leq 0 \end{cases} \quad (7)\end{aligned}$$

where

$$|0\rangle = \exp(-z_\alpha \bar{z}_\alpha) \quad a_\alpha |0\rangle = b_\alpha |0\rangle = 0 \quad \alpha = 1, 2. \quad (8)$$

Then the linear shell of the vectors $|n_1 n_2 \mathbf{m}\rangle$ may be considered as a subspace in the Fock space of a bosonic system of four degrees of freedom; this subspace is defined by the constraint

$$(a_\alpha a_\alpha^\dagger - b_\alpha b_\alpha^\dagger)|_{\text{phys}} = 0 \quad (9)$$

which, as can be readily seen, coincides with (4). We denote this subspace as H_{phys} .

From (7) it follows that the following representation of the algebra $so(4, 2)$

$$\begin{aligned}
L_{ij} &= \frac{1}{2}(a^\dagger \sigma_k a + b^\dagger \sigma_k b) & L_{i5} &= -\frac{1}{2}(a^\dagger \sigma_i C b^\dagger - a C \sigma_i b) \\
L_{i0} &= \frac{1}{2i}(a^\dagger \sigma_i C b^\dagger + a C \sigma_i b) & L_{50} &= \frac{1}{2}(a^\dagger a + b^\dagger b + 2) \\
L_{i6} &= -\frac{1}{2}(a^\dagger \sigma_i a - b^\dagger \sigma_i b) & L_{56} &= (i/2)(a^\dagger C b^\dagger - a C b) \\
L_{60} &= \frac{1}{2}(a^\dagger C b^\dagger + a C b)
\end{aligned} \tag{10}$$

may be defined in H_{phys} , where $C = i\sigma_2$ and the generators L_{AB} , $A, B, \dots = 0, \dots, 3, 5, 6$ obey commutation relations

$$[L_{AB}, L_{CD}] = i(\eta_{AD} L_{BC} + \eta_{BC} L_{AD} - \eta_{AC} L_{BD} - \eta_{BD} L_{AC}) \tag{11}$$

where $\eta_{AB} = \text{diag}(+1, -1, -1, -1, +1, -1)$. Then from (7) and (10) it follows that

$$L_{50} |n_1 n_2 m\rangle = (n_1 + n_2 + |m| + 1) |n_1 n_2 m\rangle. \tag{12}$$

Substituting (6) into (10) we obtain the following expressions for generators L_{AB} in the configuration space:

$$\begin{aligned}
L_{ij} &= \varepsilon_{ijk} (\mathbf{x} \times \mathbf{p})_k & L_{i6} &= -\frac{1}{2} x_i p^2 + p_i (\mathbf{x} \mathbf{p}) + \frac{1}{2} x_i \\
L_{i5} &= -\frac{1}{2} x_i p^2 + p_i (\mathbf{x} \mathbf{p}) - \frac{1}{2} x_i & L_{65} &= (\mathbf{x} \mathbf{p}) - i \\
L_{i0} &= -r p_i & L_{60} &= \frac{1}{2} (r p^2 - r) & L_{50} &= \frac{1}{2} (r p^2 + r).
\end{aligned} \tag{13}$$

The generators $L_{\mu\nu}$ induce the Lorentz transforms of four-vectors $n_{\mathbf{x}}^\mu$.

3. Discrete spectrum

3.1. The $Sp(2, \mathbb{R})/U(2)$ space

Let us consider a set of complex symmetric 2×2 matrices satisfying the condition

$$I - \Lambda \Lambda^\dagger > 0. \tag{14}$$

On these matrices we can define the action of the $Sp(2, \mathbb{R})$ group so that they become a symmetric space

$$SO(3, 2)/(SO(3) \otimes SO(2)) \simeq Sp(2, \mathbb{R})/U(2).$$

This space has been considered in detail previously (see, e.g., [29]). We now introduce a three-vector \mathbf{u} as $\Lambda = c\sigma\mathbf{u}$. Then (14) is equivalent to the conditions

$$|\mathbf{u}^2| < 1 \quad 1 - 2\mathbf{u}\mathbf{u}^* + \mathbf{u}^2\mathbf{u}^{*2} > 0. \tag{15}$$

The infinitesimal operators corresponding to the action of the $SO(3, 2)$ group over this space are given by [29]

$$\begin{aligned}
L_{ij} &= i \left(u_i \frac{\partial}{\partial u_j} - u_j \frac{\partial}{\partial u_i} \right) & L_{50} &= \mathbf{u} \frac{\partial}{\partial \mathbf{u}} \\
L_{5i} &= i \left(\frac{1 + \mathbf{u}^2}{2} \frac{\partial}{\partial u_i} - u_i \left(\mathbf{u} \frac{\partial}{\partial \mathbf{u}} \right) \right) \\
L_{0i} &= - \left(\frac{1 - \mathbf{u}^2}{2} \frac{\partial}{\partial u_i} + u_i \left(\mathbf{u} \frac{\partial}{\partial \mathbf{u}} \right) \right).
\end{aligned} \tag{16}$$

Let us introduce a unit complex four-vector as

$$k_{\mathbf{u}}^\mu = \left(\frac{1 + \mathbf{u}^2}{1 - \mathbf{u}^2}, \frac{2\mathbf{u}}{1 - \mathbf{u}^2} \right) \quad k_{\mathbf{u}} \cdot k_{\mathbf{u}} = 1. \tag{17}$$

Then we can rewrite the conditions (15) as

$$w_{\mathbf{u}}^0 > 0 \quad w_{\mathbf{u}} \cdot w_{\mathbf{u}} = \frac{1 - 2\mathbf{u}\mathbf{u}^* + \mathbf{u}^2\mathbf{u}^{*2}}{|1 + \mathbf{u}^2|^2} > 0 \tag{18}$$

where $w_{\mathbf{u}}^\mu = \text{Re } k_{\mathbf{u}}^\mu$. The action of generators $L_{\mu\nu}$ (16) corresponds to the Lorentz transformations of the vector $k_{\mathbf{u}}^\mu$

3.2. Coherent states

Let \mathbf{u} be a complex three-vector having the components

$$\mathbf{u} = \left(\frac{i}{2}(\lambda_2 - \lambda_1), \frac{1}{2}(\lambda_1 + \lambda_2), 0 \right) \quad (19)$$

and satisfying the conditions (15). We now construct the superposition of states

$$|\mathbf{u}\rangle = c_0 \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} (\lambda_1 \lambda_2)^{\frac{1}{2}(2n+|m|+1)} \left(\frac{\lambda_1}{\lambda_2} \right)^{m/2} |n n m\rangle. \quad (20)$$

Using the formulas [30]

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{n!}{\Gamma(n + \alpha + 1)} L_n^\alpha(x) L_n^\alpha(y) z^n \\ &= (1 - z)^{-1} \exp\left(-z \frac{x + y}{1 - z}\right) (-xy z)^{-\alpha/2} J_\alpha\left(2 \frac{(-xy z)^{1/2}}{1 - z}\right) \quad |z| < 1 \\ & \sum_{n=-\infty}^{\infty} t^n J_n(z) = \exp\left[(t - t^{-1})z/2\right] \end{aligned} \quad (21)$$

we obtain

$$\langle \mathbf{x} | \mathbf{u} \rangle = \frac{c_0}{2\sqrt{\pi}} (k_{\mathbf{u}}^2)^{1/2} \exp(-k_{\mathbf{u}} \cdot n_{\mathbf{x}}). \quad (22)$$

It can be readily seen that an arbitrary three-vector satisfying conditions (15) can be obtained by applying the $SO(3)$ transformations to a certain three-vector defined by (19). Due to (13) such a transformation corresponds to certain transformation in the space H_{phys} . Then the vector $|\mathbf{u}\rangle$ defined by the right-hand side of equality (22) can be represented as a superposition of vectors of the space H_{phys} for an arbitrary \mathbf{u} that satisfies the conditions (15). Then hereafter we will consider \mathbf{u} as an arbitrary element of the space $Sp(2, \mathbb{R})/U(2)$.

Let us choose the normalization constant so that $\langle \mathbf{u} | \mathbf{u} \rangle = 1$ i.e.

$$|c_0|^2 = \frac{1 - 2\mathbf{u}\mathbf{u}^* + \mathbf{u}^2\mathbf{u}^{*2}}{|\mathbf{u}^2|}.$$

Thus, both the conditions (15) are necessary; the first one is necessary for convergence of series (20) and the second one for normalizability of the resulting expression. Then using (18) we finally obtain

$$\langle \mathbf{x} | \mathbf{u} \rangle = \frac{1}{\pi^{1/2}} (w_{\mathbf{u}} \cdot w_{\mathbf{u}})^{1/2} \exp(-k_{\mathbf{u}} \cdot n_{\mathbf{x}}). \quad (23)$$

CS for the 1D HA constructed in [8] may be easily obtained as a particular case of (23) putting $\mathbf{x}^1 = \mathbf{x}^2 = k_{\mathbf{u}}^1 = k_{\mathbf{u}}^2 = 0$.

3.3. Symmetry properties

Now we show that the system $|\mathbf{u}\rangle$ is $SO(3, 2)$ -invariant. This system is obviously $SO(3, 1)$ -invariant; on the other hand, we have the equality

$$e^{i\varepsilon L_{50}} |\mathbf{u}\rangle = e^{i\varepsilon} |\mathbf{u} e^{i\varepsilon}\rangle$$

the validity of which can be proven either using (12) and (20) or in the infinitesimal form using (13) and (16). The $SO(3, 2)$ -invariance of this system follows then from the commutation relations (11).

From here it follows that the system of states $|\mathbf{u}\rangle$ is a system of the Perelomov's CS for the $SO(3, 2)$ group constructed starting from the $SO(3) \times SO(2)$ -invariant vector $|0\rangle$. This fact can be also proven directly for the following particular cases:

$$\begin{aligned} 1. \quad \Lambda &= \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix} \\ 2. \quad \Lambda &= \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \quad \alpha_1 \alpha_2 = 0. \end{aligned} \quad (24)$$

To this end let us introduce the operators A_α, B_α as

$$a_\alpha = A_\alpha + iB_\alpha \quad b_\alpha = A_\alpha - iB_\alpha. \quad (25)$$

Since the matrices $C\sigma_i$ are symmetric, then from (10) it follows that generators of the $SO(3, 2)$ subgroup belonging to the $SO(4, 2)$ group may be represented as a nondegenerate linear combination of generators of the $Sp(2, \mathbb{R}) \simeq SO(3, 2)$ group:

$$\begin{aligned} X_{\alpha\beta} &= A_\alpha A_\beta + B_\alpha B_\beta & X_{\alpha\beta}^\dagger &= A_\alpha^\dagger A_\beta^\dagger + B_\alpha^\dagger B_\beta^\dagger \\ Y_{\alpha\beta} &= \frac{1}{2}(A_\alpha A_\beta^\dagger + A_\beta^\dagger A_\alpha) + \frac{1}{2}(B_\alpha B_\beta^\dagger + B_\beta^\dagger B_\alpha). \end{aligned} \quad (26)$$

It follows from (26) that the group $Sp(2, \mathbb{R})$ is a group of canonical transformations of operators $A_\alpha, A_\alpha^\dagger$ and $B_\alpha, B_\alpha^\dagger$ separately. Then since (8) is equivalent to

$$A_\alpha |0\rangle = B_\alpha |0\rangle = 0$$

then we can use the analogy with the usual CS for a bosonic system of two degrees of freedom to find that the following equalities are valid:

$$(A_\alpha - \Lambda_{\alpha\beta} A_\beta^\dagger) |\Lambda\rangle = (B_\alpha - \Lambda_{\alpha\beta} B_\beta^\dagger) |\Lambda\rangle = 0. \quad (27)$$

Denote as \mathcal{H}_A the Hilbert space of states of bosonic system of two degrees of freedom composed by the vectors of the form $A_{\alpha_1}^\dagger \dots A_{\alpha_n}^\dagger |0\rangle_A$, where $A_\alpha |0\rangle_A = 0$, $\alpha = 1, 2$. Following [15], define in \mathcal{H}_A the CS system for the space $Sp(2, \mathbb{R})/U(2)$ as

$$|\Lambda\rangle_A = [\det(I - \Lambda \Lambda^\dagger)]^{1/4} \exp\left(\frac{1}{2} \Lambda_{\alpha\beta} A_\alpha^\dagger A_\beta^\dagger\right) |0\rangle_A.$$

The space \mathcal{H}_B and its CS $|\Lambda\rangle_B$ may be defined in the completely analogous way. Then we can consider the representation (26) of the $SO(3, 2)$ group as acting in the subspace of the space $\mathcal{H}_A \times \mathcal{H}_B$ defined by the constraint (9). Consider in the space $\mathcal{H}_A \times \mathcal{H}_B$ the system of states

$$|\Lambda\rangle = |\Lambda\rangle_A \otimes |\Lambda\rangle_B = [\det(I - \Lambda \Lambda^\dagger)]^{1/2} \exp\left(\frac{1}{2} \Lambda_{\alpha\beta} X_{\alpha\beta}^\dagger\right) |0\rangle \quad (28)$$

where $X_{\alpha\beta}$ are given by (26). Using (27) it is easy to show that the vectors $|\Lambda\rangle$ obey the constraint (9) and then belong to $\mathcal{H}_{\text{phys}}$. From here it follows that (28) is true Perelomov HA CS system for the space $Sp(2, \mathbb{R})/U(2)$ and thus must coincide with (23). Using the equality (7) above and the formula [30]

$$\sum_{n=0}^{\infty} L_n^\alpha(x) z^n = (1-z)^{-\alpha-1} \exp\left(\frac{xz}{z-1}\right) \quad |z| < 1$$

we can directly show that (28) indeed coincides with (23) to within a phase multiplier if Λ has the form (24).

3.4. Robertson relations

Introduce the Hermitean operators Q_a , $a = 1, \dots, 8$ as

$$\begin{aligned} Q_\alpha &= \xi_\alpha & Q_{\alpha+2} &= -i \frac{\partial}{\partial \xi_\alpha} \\ Q_{\alpha+4} &= \eta_\alpha & Q_{\alpha+6} &= -i \frac{\partial}{\partial \eta_\alpha} \end{aligned}$$

and define their dispersion in the given state as

$$\Sigma_{ab} = \frac{1}{2} \langle Q_a Q_b + Q_b Q_a \rangle - \langle Q_a \rangle \langle Q_b \rangle.$$

By the virtue of (1), (6), (25) and (27) the operators Q_a acting on the vectors $|\mathbf{u}\rangle$ satisfy $4 = 8/2$ linearly independent equalities. The Robertson inequality for the dispersion of a set of Hermitian operators

$$\det \Sigma \geq \det \Omega \quad \Omega_{ab} = -\frac{i}{2} \langle [Q_a, Q_b] \rangle$$

is then transformed into the equality, in view of the results of [31] if the mean values are determined for an arbitrary CS $|\mathbf{u}\rangle$.

4. Continuous spectrum

4.1. The $SO(4, 1)$ group

Let us introduce the generators

$$\Pi_i^\pm = L_{6i} \pm L_{0i}.$$

The generators Π^+ and Π^- form two Abelian subgroups, which we denote by \mathcal{T}^+ and \mathcal{T}^- ; the subgroups induced by the generators L_{06} and L_{ij} we denote as \mathcal{T}^0 and \mathcal{R} respectively. Finite transformations are denoted as

$$\Theta_\pm(\mathbf{A}) = \exp(i\Pi^\pm \mathbf{A}) \quad \Theta_0(\varepsilon) = \exp(iL_{06}\varepsilon).$$

Consider the action $\mathbf{v} \mapsto \mathbf{v}_g$ of elements $g \in SO(4, 1)$ over the vectors $\mathbf{v} \in \mathbb{R}^3$ defined by

$$\begin{aligned} g = \Theta_-(\mathbf{A}) : \mathbf{v}_g &= \mathbf{v} - \mathbf{A} \\ g = \Theta_+(\mathbf{A}) : \mathbf{v}_g &= \frac{\mathbf{v} + \mathbf{A}v^2}{1 + 2\mathbf{v}\mathbf{A} + v^2\mathbf{A}^2} \\ g = \Theta_0(\varepsilon) : \mathbf{v}_g &= \mathbf{v}e^\varepsilon. \end{aligned} \tag{29}$$

Generators have the form

$$\begin{aligned} i\Pi^- &= -\frac{\partial}{\partial \mathbf{v}} & i\Pi^+ &= v^2 \frac{\partial}{\partial \mathbf{v}} - 2\mathbf{v} \left(\mathbf{v} \frac{\partial}{\partial \mathbf{v}} \right) \\ iL_{06} &= \mathbf{v} \frac{\partial}{\partial \mathbf{v}} & iL_{ik} &= v_k \frac{\partial}{\partial v_i} - v_i \frac{\partial}{\partial v_k}. \end{aligned} \tag{30}$$

The stationary subgroup of the point $\mathbf{v} = \mathbf{0}$ is $\mathcal{K} = \mathcal{T}^+(\mathcal{T}^0 \otimes \mathcal{R})$; then the space \mathbb{R}^3 equipped with such an action of the $SO(4, 1)$ group may be identified with the coset space $SO(4, 1)/\mathcal{K}$. Then the action of the group $SO(4, 1)$ on the unit real four-vector $\mathbf{k}_\mathbf{v}$ (17) is defined; the generators $L_{\mu\nu}$ correspond to the Lorentz transformations of this vector.

Concerning the conformal action of orthogonal groups over euclidean and pseudoeuclidean spaces see also [32].

4.2. Wave functions

In the case of positive energy we can use the coordinate stretching to reduce equations (3),(4) to the Schrodinger equation for two ‘oscillators’ having unit mass, frequency equal to i, and the same values of angular momentum. Then the solutions of equation (2) corresponding to energy $E = [2(\rho_1 + \rho_2)^2]^{-1}$ are

$$\Psi_{\rho_1 \rho_2 m}(\mathcal{X}) = \psi_{\rho_1 \rho_2 m}(\mathcal{X}(r_0(\rho_1 + \rho_2))^{-1})$$

where

$$\begin{aligned} \psi_{\rho_1 \rho_2 m}(\mathcal{X}) &\equiv |\rho_1 \rho_2 m\rangle = e^{im\phi} \varphi_{\rho_1 |m|}(\xi) \varphi_{\rho_2 |m|}(\eta) \\ \varphi_{\rho |m|}(\xi) &= (2\pi i e^{\pi\rho})^{-1/2} \left| \Gamma\left(-i\rho + \frac{|m|+1}{2}\right) \right| \\ &\times e^{i\xi^2/2} (-i\xi^2)^{|m|/2} {}_1F_1\left(-i\rho + \frac{|m|+1}{2}, |m|+1, -i\xi^2\right). \end{aligned}$$

To within the change of $|m|$ to $2l+1$, the functions $\varphi_{\rho |m|}$ coincide with radial components of the wave functions of the continuous spectrum of HA obtained previously in [33]. The normalization factors are chosen so that the functions $\varphi_{\rho |m|}(\xi)$ are real and satisfy the normalization conditions

$$\int_0^\infty d(\xi^2) \varphi_{\rho |m|}(\xi) \varphi_{\rho' |m|}(\xi) = \delta(\rho - \rho').$$

Then

$$\langle \rho_1 \rho_2 m | \rho'_1 \rho'_2 m' \rangle = \delta(\rho_1 - \rho'_1) \delta(\rho_2 - \rho'_2) \delta_{mm'}.$$

The equality

$$L_{06} |\rho_1 \rho_2 m\rangle = -(\rho_1 + \rho_2) |\rho_1 \rho_2 m\rangle \quad (31)$$

holds.

4.3. Coherent states

Let $\mathbf{v} = (-v \cos \theta, v \sin \theta, 0)$. By the analogy with (20) define the states $|\mathbf{v}\rangle$ as

$$|\mathbf{v}\rangle = (\mathbf{k}_{\mathbf{v}}^2)^{-1/2} \sum_{m=-\infty}^{\infty} e^{im\theta} \int_{-\infty}^{\infty} d\rho v^{-2i\rho} |\rho \rho m\rangle. \quad (32)$$

Inverting the Mellin transform of the confluent hypergeometric function [34]

$$\begin{aligned} & \int_0^{\infty} t^{s-1} dt (1+t^2)^{-1} \exp\left(-i \frac{\xi^2 + \eta^2}{2} \frac{1-t^2}{1+t^2}\right) J_{|m|}\left(-i\xi\eta \frac{2t}{1+t^2}\right) \\ &= \pi i^{|m|+1} e^{\pi\rho} \varphi_{\rho|m|}(\xi) \varphi_{\rho|m|}(\eta) \Big|_{i\rho=\frac{s-1}{2}} \end{aligned}$$

and using (21) we obtain

$$\langle \mathcal{X} | \mathbf{v} \rangle = \exp(-i \mathbf{k}_{\mathbf{v}} \cdot \mathbf{n}_{\mathcal{X}}). \quad (33)$$

Using (31) and (32) we obtain that at $g \in \mathcal{T}^0$

$$T(g) |\mathbf{v}\rangle = \left(\frac{d\mu(k\mathbf{v}_g)}{d\mu(k\mathbf{v})} \right)^{1/3} |\mathbf{v}_g\rangle \quad (34)$$

where $T(g)$ is the representation of the $SO(4, 1)$ group with the Lie algebra given by (13), and $d\mu(k) = (k^0)^{-1} d^3\mathbf{k}$ is the Lorentz-invariant measure on the hyperboloid $\mathbf{k} \cdot \mathbf{k} = 1$. The validity of equality (34) can be also proved in the infinitesimal form using (13) and (30). From the other hand, the validity of (34) at $g \in SO(3, 1)$ is obvious and then it is correct for all $g \in SO(4, 1)$.

Define the space $SO(4, 1)/(\mathcal{T}^+ \mathcal{R})$ as a set of pairs (\mathbf{v}, τ) , where $\tau \in \mathbb{R} \setminus \{0\}$ and the action of the $SO(4, 1)$ group is defined by (29) and

$$\tau_g = \left(\frac{d\mu(k\mathbf{v}_g)}{d\mu(k\mathbf{v})} \right)^{1/3} \tau.$$

Then the states

$$|\mathbf{v}\tau\rangle = \tau^{-1} |\mathbf{v}\rangle$$

compose a system of Perelomov's CS for the mentioned space.

5. Relation to the conformal group

The twistor space $SO(4, 2)/(SO(4) \otimes SO(2))$ is a domain in \mathbb{C}^4 defined by the inequalities

$$|u_a u_a| < 1 \quad 1 - 2u_a u_a^* + |u_a u_a|^2 > 0 \quad a = 1, \dots, 4. \quad (35)$$

We obtain another realization of the twistor space considering the mapping

$$z^0 = i \frac{1 + u_a u_a - 2u_4}{1 - u_a u_a + 2iu_4} \quad z^k = \frac{2iu_k}{1 - u_a u_a + 2iu_4}. \quad (36)$$

Then (35) transforms to

$$\mathfrak{Z}z^0 > 0 \quad \mathfrak{Z}z \cdot \mathfrak{Z}z > 0.$$

Consider the set of holomorphic C^∞ -functions which are square integrable over the twistor space with respect to the $SO(4, 2)$ -invariant measure $(\mathfrak{Z}z \cdot \mathfrak{Z}z)^{-4} d\mathbf{z}$. Over these functions one can define the $SO(4, 2)$ group irreducible representation belonging to the discrete series and having the generators

$$\begin{aligned} i(L_{5\mu} + L_{6\mu}) &= (z \cdot z) \frac{\partial}{\partial z^\mu} - 2z_\mu \left(z \cdot \frac{\partial}{\partial z} \right) - 2z_\mu \\ i(L_{5\mu} - L_{6\mu}) &= \frac{\partial}{\partial z^\mu} \\ iL_{\mu\nu} &= z_\mu \frac{\partial}{\partial z^\nu} - z_\nu \frac{\partial}{\partial z^\mu} \quad iL_{65} = z \cdot \frac{\partial}{\partial z} + 1. \end{aligned} \tag{37}$$

Consider the functions

$$\langle \mathcal{X} | z \rangle = e^{in\mathbf{x} \cdot \mathbf{z}}.$$

Then one can show [35] that the integral transformation

$$F(z) = \int \frac{d^3 \mathcal{X}}{r} \langle \mathcal{X} | z \rangle f(\mathcal{X}) \tag{38}$$

interwines the representations (13) and (37) of the $SO(4, 2)$ group. At the level of Lie algebras this fact can be directly observed since the difference between generators (13) and (37) vanishes acting on the functions $\langle \mathcal{X} | z \rangle$.

We can pass from the twistor space to the $SO(3, 2)/SO(3) \times SO(2)$ space letting $\mathbf{u}_4 = 0$; then (35) transforms into (15), and from (36) it follows that $z^\mu = ik_{\mathbf{u}}^\mu$. From the other hand, the functions $\langle \mathcal{X} | z \rangle$ transform into CS given by (23) to within a normalization factor.

Letting $\mathfrak{Z}z^\mu = 0$ we pass to the Shilov boundary of the twistor space which coincides with the Minkowski space. If we additionally let $z^\mu = k_{\mathbf{v}}^\mu$ then the functions $\langle \mathcal{X} | z \rangle$ pass into the states (33).

Passing to the Shilov boundary, the representation (37) transforms into the representation which describes massless spin zero particles over the Minkowski space [36]. In this case, the transformation (38) shows the coincidence of representations of the $SO(4, 2)$ group which describe the hydrogen atom and massless spin zero particles over the Minkowski space. This coincidence has been previously proven in a more complicated way in [37].

Let us consider now the manifold which belongs to the boundary of the twistor space and is defined by the equality $\mathfrak{Z}z \cdot \mathfrak{Z}z = 0$ (however, we still have $\mathfrak{Z}z^0 > 0$), and moreover we assume that $\mathfrak{Z}^2 = 0$. Then the obtained manifold is Lagrangian, and we can represent the HA wave functions in the form of an integral over this manifold of the functions $\langle \mathcal{X} | z \rangle$ with a certain weight factor [28]. This indicates the possibility of a quasi-classical description of HA in terms of CS constructed above; this question requires a further investigation.

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